Linear Algebra

Vector Space

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Table of contents

01 Review

02 Field

03 Vector Space

04 Linear Combination

05 Span-Linear Hull

01

Review

Complex Numbers

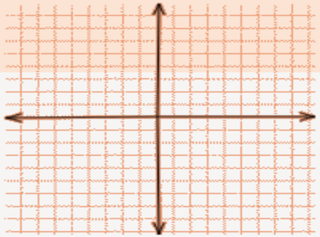
Sum and Products

Tuple and Vector Space

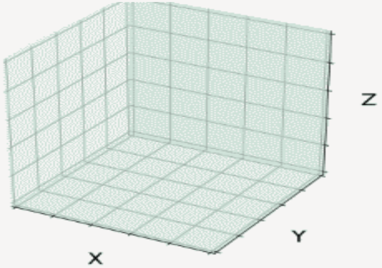
Definition

- A tuple is an ordered list of numbers.
- For example: $\begin{bmatrix} 1 \\ 2 \\ 32 \\ 10 \end{bmatrix}$ is a 4-tuple (a tuple with 4 elements).

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.112 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \pi \\ e \end{pmatrix}, \dots \right\}$$



$$\mathbb{R}^3 = \left\{ \begin{pmatrix} 17 \\ \pi \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ -2 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 22 \\ 2 \end{pmatrix}, \dots \right\}$$



Review: Complex Numbers

Numbers:

- Real: Nearly any number you can think of is a Real Number!

1	12.38	−0.8625	3/4	√2	1998

- Imaginary: When squared give a negative result.

The “unit” imaginary number (like 1 for Real Numbers) is “***i***”, which is the square root of −1.

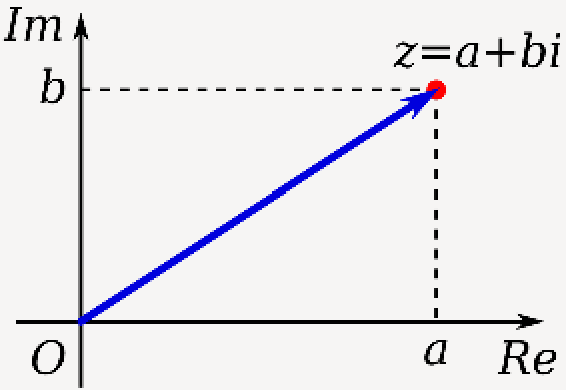
Examples of Imaginary Numbers:

3i	1.04i	−2.8i	3i/4	(√2)i	1998i

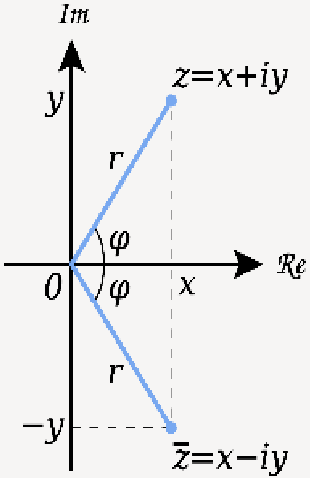
And we keep that little "i" there to remind us we need to multiply by $\sqrt{-1}$

Review: Complex Numbers

- $\mathbb{C}$ is a plane, where number $(a + bi)$ has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$
- Imaginary number: $bi\ , b \in \mathbb{R}$



- Conjugate of $x + yi$ is noted by $\overline{x + yi}$:
  - $x - yi$



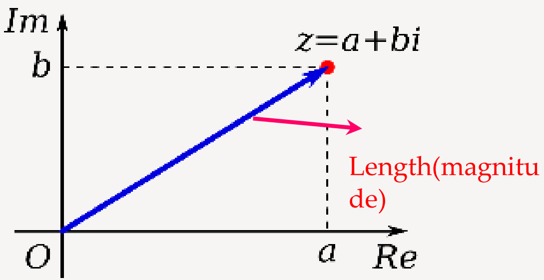
(Complex conjugate)

Review: Complex Numbers

- Arithmetic with complex numbers $(a + bi)$:
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = ac - bd + (bc + ad)i$
  - $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$

Review: Complex Numbers

- Length (magnitude): $|a+bi|^2 = \overline{(a+bi)}(a+bi) = a^2 + b^2$
- Inner Product:
  - Real: $< x, y > = x_1 y_1 + x_2 y_2 + \ ... + x_n y_n$
  - Complex: $< x, y > = \overline{x_1} y_1 + \overline{x_2} y_2 + \ ... + \overline{x_n} y_n$



Extra resource:

If you want to learn more about complex numbers, this video is recommended!

Binary Operations

What is a binary operation?

Binary Operations

Definition

Any function from A x A → A is a binary operation.

- **Closure Law:**
  - A set is said to be closure under an operation (like addition, subtraction, multiplication, etc.) if that operation is performed on elements of that set and result also lies in set.

$$if\ a \in A, b \in A \ \rightarrow a * b \in A$$

Binary Operations

Example

- ❑ Is "+" a binary operator on natural numbers?
- ❑ Is "x" a binary operator on natural numbers?
- ❑ Is "-" a binary operator on natural numbers?
- ❑ Is "/" a binary operator on natural numbers?

02

Field

Groups

Definition

A group G is a pair $(S,\circ)$, where S is a set and $\circ$ is a binary operation on S such that:

- $\circ$ is **associative**
- **(Identity)** There exists an element $e \in S$ such that:

$$e \circ a = a \circ e = a \quad \forall a \in S$$

- **(Inverses)** For every $a \in S$ there is $b \in S$ such that:

$$a \circ b = b \circ a = e$$

If $\circ$ is commutative, then G is called a commutative group!

Fields

Definition

A field F is a set together with two binary operations + and *, satisfying the following properties:

1. (F,+) is a commutative group
   - Associative
   - Identity
   - Inverses
   - Commutative
2. (F-{0},*) is a commutative group
3. The distributive law holds in F:

$$(a+b)*c=(a*c)+(b*c)$$

$$a*(b+c)=(a*b)+(a*c)$$

Fields

- A field in mathematics is a set of things of elements (not necessarily numbers) for which the basic arithmetic operations (addition, subtraction, multiplication, division) are defined: (F,+,.)

Example

$(\mathbb{R}; +, .)$ and $(\mathbb{Q}; +, .)$ serve as examples of fields.

- Field is a set (F) with two binary operations (+ , .) satisfying following properties:

Fields

$\forall$ **a, b, c** $\in \boldsymbol{F}$

Properties	Binary Operations	
	Addition (+)	**Multiplication (.)**
Closure (بسته بودن)	$\exists a + b \in F$	$\exists a.b \in F$
Associative (شرکت‌پذیری)	$a + (b + c) = (a + b) + c$	$a.(b.c) = (a.b).c$
Commutative (جابه‌جایی‌پذیری)	$a + b = b + a$	$a.b = b.a$
Existence of identity $e \in F$	$a + e = a = e + a$	$a.e = a = e.a$
Existence of inverse: For each a in F there must exist b in F	$a + b = e = b + a$	$a.b = e = b.a$ *For any nonzero a*

Multiplication is distributive over addition

$$a.(b + c) = a.b + a.c$$

$$(a + b).c = a.c + b.c$$

Fields

Example

Set $B = \{0,1\}$ under following operations is a field?

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Fields

Example

Which are fields? (two binary operations + , *)

$\mathbb{R}$

$\mathbb{C}$

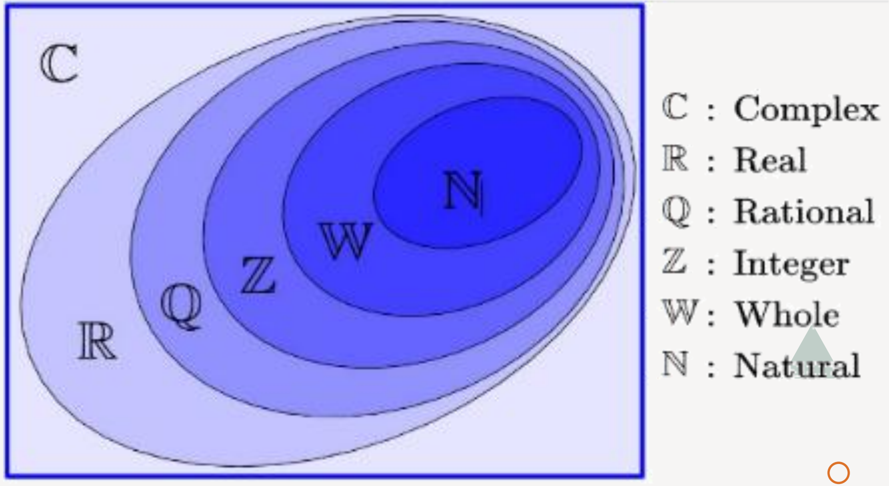
$\mathbb{Q}$

$\mathbb{Z}$

W

$\mathbb{N}$

$\mathbb{R}^{2\times 2}$



03

Vector Space

Vector Space

- Building blocks of linear algebra.
- A non-empty set V with field F (most of time R or C) forms a vector space with two operations:
  1. + : Binary operation on V which is $V \times V \rightarrow V$
  2. . : $F \times V \rightarrow V$

Note

In our course, by **default**, field is **R** (real numbers).

Vector Space

Definition

A vector space over a field F is the set V equipped with two operations: $(V, F, +, .)$

i. Vector addition: denoted by "+" adds two elements $x, y \in V$ to produce another element $x + y \in V$

ii. Scalar multiplication: denoted by "." multiplies a vector $x \in V$ with a scalar $\alpha \in F$ to produce another vector $\alpha . x \in V$. We usually omit the "." and simply write this vector as αx.

Vector Space Properties

- Addition of vector space $(x + y)$

  - **Commutative** $x + y = y + x\ \forall x, y \in V$
  - **Associative** $(x + y) + z = x + (y + z)\ \forall x, y, z \in V$
  - **Additive identity** $\exists \mathbf{0} \in V$ such that $x + \mathbf{0} = x, \forall x \in V$
  - **Additive inverse** $\exists (-x) \in V$ such that $x + (-x) = 0, \forall x \in V$

Vector Space Properties

- Action of the scalars field on the vector space (αx)
  - **Associative** $\alpha(\beta x) = (\alpha\beta)x$ $\forall \alpha, \beta \in F; \forall x \in V$
  - **Distributive over** ……

    scalar addition: $(\alpha + \beta)x = \alpha x + \beta x$ $\forall \alpha, \beta \in F; \forall x \in V$

    vector addition: $\alpha(x + y) = \alpha x + \alpha y$ $\forall \alpha \in F; \forall x, y \in V$
  - **Scalar identity** $1x = x$ $\forall x \in V$

Vector Space

Theorem

Every vector space has a unique additive identity.

Every $v \in V$ has a unique additive inverse.

Proof

Vector Space

Example

Let V be the set of all real numbers with the operations $u \oplus v = u - v$, $\oplus$ is an ordinary subtraction) and $c \boxdot u = cu$ ($\boxdot$ is an ordinary multiplication).
Is V a vector space? If it's not, which properties fail to hold?

Vector Space

Example: Fields are R in this example:

- The n-tuple space,
- The space of m x n matrices
- The space of functions:

  (f + g)(x) = f(x) + g(x) and (cf)(x) = cf(x)

  f(t) = 1 + sin(2t) and g(t) = 2 + 0.5t

- The space of polynomial functions over a field F:

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \ \ldots + a_n t^n$$

Vector Space of functions

- Function addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x) \quad and \; (af)(x) = af(x)$$

Non-empty set X and any field F → $F^x = \{f: X \to F\}$

Example

- Set of all polynomials with real coefficients
- Set of all real-valued continuous function on [0,1]
- Set of all real-valued function that are differentiable on [0,1]

Vector Space of polynomials

$P_n(\mathbb{R})$: Polynomials with max degree (n)

- Vector addition
- Scalar multiplication
- And other 8 properties!

Vector Space

Example

Which are vector spaces with + , *?

- ❑ Set $\mathbb{R}^n$ over $\mathbb{R}$
- ❑ Set $\mathbb{C}$ over $\mathbb{R}$
- ❑ Set $\mathbb{R}$ over $\mathbb{C}$
- ❑ Set $\mathbb{Z}$ over $\mathbb{R}$
- ❑ Set of all polynomials with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- ❑ Set of all polynomials of degree at most n with coefficient from $\mathbb{R}$ over $\mathbb{R}$
- ❑ Matrix: $M_{m,n}(\mathbb{R})$ over $\mathbb{R}$
- ❑ Function: $f(x): x \longrightarrow \mathbb{R}$ over $\mathbb{R}$

Conclusion

The operations on field F are:

- $+ : F \times F \rightarrow F$
- $\times : F \times F \rightarrow F$

The operations on a vector space V over a field F are:

- $+ : V \times V \rightarrow V$
- $. : F \times V \rightarrow V$

04

Linear Combination

Linear Combinations

- The linear combinations of m vectors $a_1, \dots a_m$, each with size n is:

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

where $\beta_1, \dots, \beta_m$ are scalars and called the coefficients of the linear combination

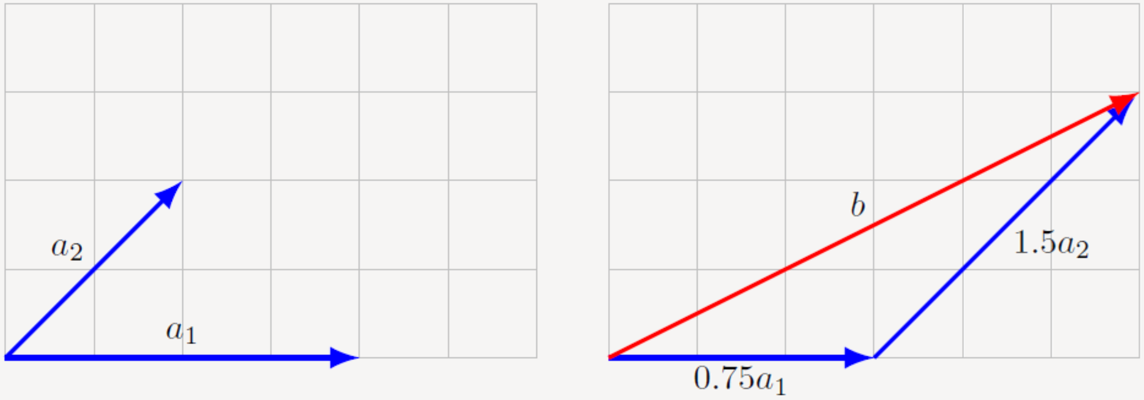
- **Coordinates**: We can write any n-vector b as a linear combination of the standard unit vectors, as:

$$b = b_1 e_1 + \cdots + b_n e_n$$

  - Example: What are the coefficients and combination for this vector?

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

Linear Combinations



Left. Two 2-vectors a_1 and a_2. *Right.* The linear combination $b = 0.75a_1 + 1.5a_2$

Special Linear Combinations

- Sum of vectors
- Average of vectors

05

Span – Linear Hull

Span or linear hull

Definition

If $v_1, v_2, v_3, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2, \ldots, v_p$ is denoted by Span $\{v_1, v_2, \ldots, v_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned (or generated)** by $v_1, v_2, \ldots, v_p$.

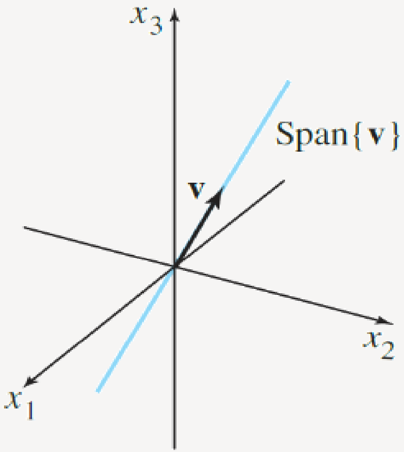
That is, Span$\{v_1, v_2, \ldots, v_p\}$ is the collection of all vectors that can be written in the form:

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p$$

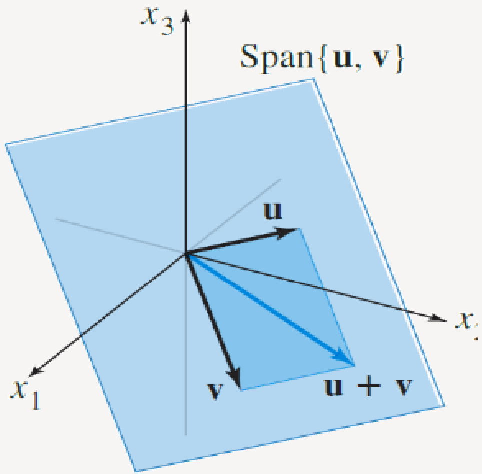
with $c_1, c_2, \ldots, c_p$ being scalars.

Span Geometry

v and u are non-zero vectors in $\mathbb{R}^3$ where v is not a multiple of u

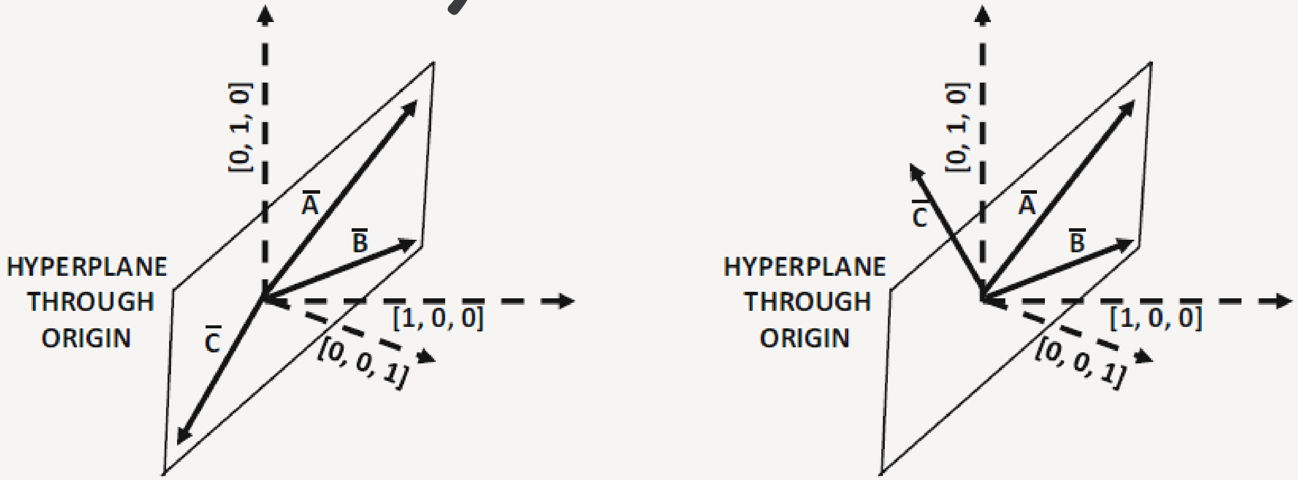


Span {**v**} as a
line through the origin.



Span {**u**, **v**} as a
plane through the origin.

Span Geometry



(a) $\text{Span}(\{\overline{A}, \overline{B}\}) = \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors on hyperplane

(b) $\text{Span}(\{\overline{A}, \overline{B}\}) \neq \text{Span}(\{\overline{A}, \overline{B}, \overline{C}\})$
$\text{Span}(\{\overline{A}, \overline{B}, \overline{C}\}) =$ All vectors in $\mathcal{R}^3$

Figure 2.6: The span of a set of linearly dependent vectors has lower dimension than the number of vectors in the set

Span or linear hull

Example

- Is vector b in Span $\{v_1, v_2, \dots, v_p\}$
- Is vector v_3 in Span $\{v_1, v_2, \dots, v_p\}$
- Is vector 0 in Span $\{v_1, v_2, \dots, v_p\}$
- Span of polynomials: $\{(1+x), (1-x), x^2\}$?
- Is b in Span $\{a_1, a_2\}$?

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Resources

- Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.
- David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016.
- Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.